



11076CH05

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ *Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics.* – GAUSS ❖

4.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.



W. R. Hamilton
(1805-1865)

4.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol i . Then, we have $i^2 = -1$. This means that i is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by $\text{Re } z$ and b is called the *imaginary part* denoted by $\text{Im } z$ of the complex number z . For example, if $z = 2 + i5$, then $\text{Re } z = 2$ and $\text{Im } z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Example 1 If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y .

Solution We have

$$4x + i(3x - y) = 3 + i(-6) \quad \dots (1)$$

Equating the real and the imaginary parts of (1), we get

$$4x = 3, 3x - y = -6,$$

which, on solving simultaneously, give $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

4.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

4.3.1 Addition of two complex numbers Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$$z_1 + z_2 = (a + c) + i(b + d), \text{ which is again a complex number.}$$

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i8$

The addition of complex numbers satisfy the following properties:

- (i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2 .
- (ii) *The commutative law* For any two complex numbers z_1 and z_2 , $z_1 + z_2 = z_2 + z_1$
- (iii) *The associative law* For any three complex numbers z_1, z_2, z_3 , $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
- (iv) *The existence of additive identity* There exists the complex number $0 + i0$ (denoted as 0), called the *additive identity* or the *zero complex number*; such that, for every complex number z , $z + 0 = z$.
- (v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of z* . We observe that $z + (-z) = 0$ (the additive identity).

4.3.2 Difference of two complex numbers Given any two complex numbers z_1 and z_2 , the difference $z_1 - z_2$ is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example, $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

4.3.3 Multiplication of two complex numbers Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

For example, $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2 .

(ii) **The commutative law** For any two complex numbers z_1 and z_2 ,

$$z_1 z_2 = z_2 z_1$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3 ,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i0$ (denoted as 1), called the *multiplicative identity* such that $z \cdot 1 = z$, for every complex number z .

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi$ ($a \neq 0, b \neq 0$), we have the complex number

$$\frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2} \text{ (denoted by } \frac{1}{z} \text{ or } z^{-1} \text{), called the } \textit{multiplicative inverse}$$

of z such that

$$z \cdot \frac{1}{z} = 1 \text{ (the multiplicative identity).}$$

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3 ,

$$(a) \quad z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

$$(b) \quad (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$$

4.3.4 Division of two complex numbers Given any two complex numbers z_1 and z_2 ,

where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

$$\text{Then } \frac{z_1}{z_2} = \left((6 + 3i) \times \frac{1}{2 - i} \right) = (6 + 3i) \left(\frac{2}{2^2 + (-1)^2} + i \frac{-(-1)}{2^2 + (-1)^2} \right)$$

$$= (6 + 3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}[12 - 3 + i(6+6)] = \frac{1}{5}(9 + 12i)$$

4.3.5 Power of i we know that

$$i^3 = i^2 i = (-1) i = -i, \quad i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^5 = (i^2)^2 i = (-1)^2 i = i, \quad i^6 = (i^2)^3 = (-1)^3 = -1, \text{ etc.}$$

Also, we have
$$i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k , $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

4.3.6 The square roots of a negative real number

Note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Therefore, the square roots of -1 are $i, -i$. However, by the symbol $\sqrt{-1}$, we would mean i only.

Now, we can see that i and $-i$ both are the solutions of the equation $x^2 + 1 = 0$ or $x^2 = -1$.

Similarly
$$(\sqrt{3}i)^2 = (\sqrt{3})^2 i^2 = 3(-1) = -3$$

$$(-\sqrt{3}i)^2 = (-\sqrt{3})^2 i^2 = -3$$

Therefore, the square roots of -3 are $\sqrt{3}i$ and $-\sqrt{3}i$.

Again, the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}i$ only, i.e., $\sqrt{-3} = \sqrt{3}i$.

Generally, if a is a positive real number, $\sqrt{-a} = \sqrt{a} \sqrt{-1} = \sqrt{a}i$,

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real number a and b . This result also holds true when either $a > 0, b < 0$ or $a < 0, b > 0$. What if $a < 0, b < 0$? Let us examine.

Note that

$$i^2 = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)} \text{ (by assuming } \sqrt{a} \times \sqrt{b} = \sqrt{ab} \text{ for all real numbers)}$$

$$= \sqrt{1} = 1, \text{ which is a contradiction to the fact that } i^2 = -1.$$

Therefore, $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers.

Further, if any of a and b is zero, then, clearly, $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

4.3.7 Identities We prove the following identity

$$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2, \text{ for all complex numbers } z_1 \text{ and } z_2.$$

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$,

$$= (z_1 + z_2)z_1 + (z_1 + z_2)z_2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2 \quad \text{(Commutative law of multiplication)}$$

$$= z_1^2 + 2z_1z_2 + z_2^2$$

Similarly, we can prove the following identities:

$$(i) \quad (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$$

$$(ii) \quad (z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$$

$$(iii) \quad (z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$$

$$(iv) \quad z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

$$(i) \quad (-5i) \left(\frac{1}{8}i \right) \qquad (ii) \quad (-i)(2i) \left(-\frac{1}{8}i \right)^3$$

Solution (i) $(-5i) \left(\frac{1}{8}i \right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

$$(ii) \quad (-i)(2i) \left(-\frac{1}{8}i \right)^3 = 2 \times \frac{1}{8 \times 8 \times 8} \times i^5 = \frac{1}{256}(i^2)^2 i = \frac{1}{256}i.$$

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5 (3i)^2 - (3i)^3$
 $= 125 - 225i - 135 + 27i = -10 - 198i$.

Example 4 Express $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$ in the form of $a + ib$

Solution We have, $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i) = (-\sqrt{3} + \sqrt{2}i)(2\sqrt{3} - i)$
 $= -6 + \sqrt{3}i + 2\sqrt{6}i - \sqrt{2}i^2 = (-6 + \sqrt{2}) + \sqrt{3}(1 + 2\sqrt{2})i$

4.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z , denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2 + b^2}$, i.e., $|z| = \sqrt{a^2 + b^2}$ and the conjugate of z , denoted as \bar{z} , is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $|3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$, $|2 - 5i| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$,

and $\overline{3 + i} = 3 - i$, $\overline{2 - 5i} = 2 + 5i$, $\overline{-3i - 5} = 3i - 5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a + ib} = \frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

or $z \bar{z} = |z|^2$

Furthermore, the following results can easily be derived.

For any two complex numbers z_1 and z_2 , we have

(i) $|z_1 z_2| = |z_1| |z_2|$ (ii) $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of $2 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$\begin{aligned} z^{-1} &= \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)} \\ &= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i \end{aligned}$$

Example 6 Express the following in the form $a + ib$

(i) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Solution (i) We have, $\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} = \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-(\sqrt{2}i)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i.$$

(ii) $i^{-35} = \frac{1}{i^{35}} = \frac{1}{(i^2)^{17}i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = i$

EXERCISE 4.1

Express each of the complex number given in the Exercises 1 to 10 in the form $a + ib$.

1. $(5i)\left(-\frac{3}{5}i\right)$

2. $i^9 + i^{19}$

3. i^{-39}

4. $3(7 + i7) + i(7 + i7)$ 5. $(1 - i) - (-1 + i6)$
 6. $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$ 7. $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$
 8. $(1 - i)^4$ 9. $\left(\frac{1}{3} + 3i\right)^3$ 10. $\left(-2 - \frac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$ 12. $\sqrt{5} + 3i$ 13. $-i$
 14. Express the following expression in the form of $a + ib$:

$$\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

4.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y) , we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa.

Some complex numbers such as $2 + 4i, -2 + 3i, 0 + 1i, 2 + 0i, -5 - 2i$ and $1 - 2i$ which correspond to the ordered

pairs $(2, 4), (-2, 3), (0, 1), (2, 0), (-5, -2)$, and $(1, -2)$, respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 4.1.

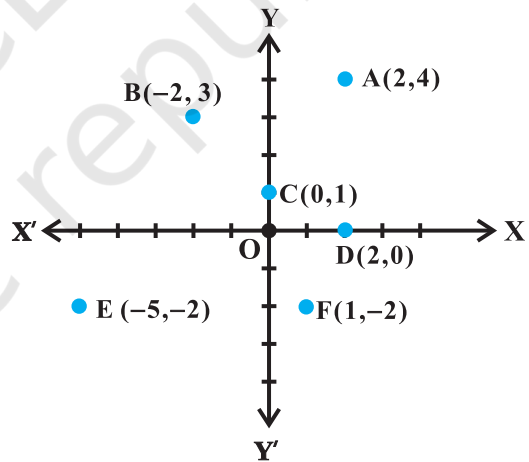


Fig 4.1

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

Obviously, in the Argand plane, the modulus of the complex number $x + iy = \sqrt{x^2 + y^2}$ is the distance between the point $P(x, y)$ and the origin $O(0, 0)$ (Fig 4.2). The points on the x -axis corresponds to the complex numbers of the form $a + i 0$ and the points on the y -axis corresponds to the complex numbers of the form

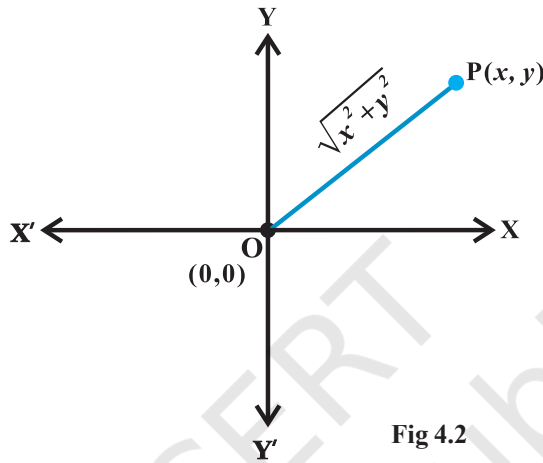


Fig 4.2

$0 + i b$. The x -axis and y -axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $z = x - iy$ in the Argand plane are, respectively, the points $P(x, y)$ and $Q(x, -y)$.

Geometrically, the point $(x, -y)$ is the mirror image of the point (x, y) on the real axis (Fig 4.3).

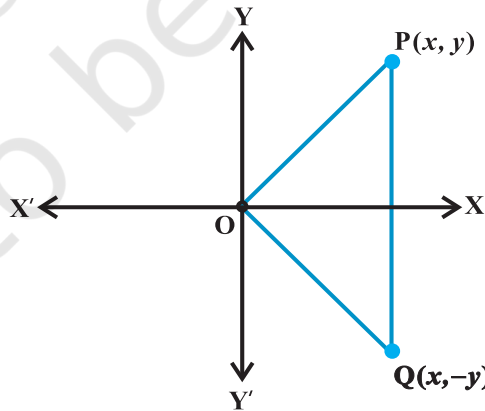


Fig 4.3

Miscellaneous Examples

Example 7 Find the conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have ,
$$\begin{aligned} & \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} \\ &= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} \\ &= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i \end{aligned}$$

Therefore, conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\frac{63}{25} + \frac{16}{25}i$.

Example 8 If $x + iy = \frac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2 - b^2 + 2abi}{a^2 + b^2} = \frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$$

So that, $x - iy = \frac{a^2 - b^2}{a^2 + b^2} - \frac{2ab}{a^2 + b^2}i$

Therefore,

$$x^2 + y^2 = (x + iy)(x - iy) = \frac{(a^2 - b^2)^2}{(a^2 + b^2)^2} + \frac{4a^2b^2}{(a^2 + b^2)^2} = \frac{(a^2 + b^2)^2}{(a^2 + b^2)^2} = 1$$

Miscellaneous Exercise on Chapter 4

1. Evaluate: $\left[i^{18} + \left(\frac{1}{i} \right)^{25} \right]^3$.

2. For any two complex numbers z_1 and z_2 , prove that $\text{Re}(z_1 z_2) = \text{Re} z_1 \text{Re} z_2 - \text{Im} z_1 \text{Im} z_2$

3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form .
4. If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$.
5. If $z_1 = 2 - i, z_2 = 1 + i$, find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$.
6. If $a + ib = \frac{(x+i)^2}{2x^2 + 1}$, prove that $a^2 + b^2 = \frac{(x^2 + 1)^2}{(2x^2 + 1)^2}$.
7. Let $z_1 = 2 - i, z_2 = -2 + i$. Find
- (i) $\operatorname{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\operatorname{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$.
8. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.
9. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.
10. If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$.
11. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.
12. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.
13. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that $(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$
14. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m .

Summary

- ◆ A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary part* of the complex number.
- ◆ Let $z_1 = a + ib$ and $z_2 = c + id$. Then
 - (i) $z_1 + z_2 = (a + c) + i(b + d)$
 - (ii) $z_1 z_2 = (ac - bd) + i(ad + bc)$
- ◆ For any non-zero complex number $z = a + ib$ ($a \neq 0, b \neq 0$), there exists the complex number $\frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}$, denoted by $\frac{1}{z}$ or z^{-1} , called the *multiplicative inverse* of z such that $(a + ib) \left(\frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2} \right) = 1 + i0$
 $= 1$
- ◆ For any integer k , $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$
- ◆ The conjugate of the complex number $z = a + ib$, denoted by \bar{z} , is given by $\bar{z} = a - ib$.

Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850) who first stated this difficulty clearly. “He mentions in his work ‘*Ganitasara Sangraha*’ as in the nature of things a negative (quantity) is not a square (quantity)”, it has, therefore, no square root”. *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. “There is no square root of a negative quantity, for it is not a square.” *Cardan* (1545) considered the problem of solving

$$x + y = 10, xy = 40.$$

He obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$ as the solution of it, which was discarded by him by saying that these numbers are ‘useless’. *Albert Girard* (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol i for $\sqrt{-1}$ and *W.R. Hamilton* (about 1830) regarded the complex number $a + ib$ as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called ‘*imaginary numbers*’.

